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1987 J. Phys. A: Math. Gen. 20 L9

(http://iopscience.iop.org/0305-4470/20/1/002)

J. Phys. A: Math. Gen. **20** (1987) L9–L11. Printed in the UK

LETTER TO THE EDITOR

Bipartitioning of random graphs of fixed extensive valence

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Received 7 August 1986, in final form 14 October 1986

Abstract. The procedure of Fu and Anderson for the application of statistical mechanics to the problem of bipartitioning random graphs is extended to graphs of fixed extensive valence. The cost function is shown to be independent of whether the valence is locally or globally constrained.

In a recent paper Fu and Anderson (1986, hereafter referred to as FA) have applied techniques of the statistical mechanics of random systems to graph partitioning (Palmer 1985). In the class of problems they considered, each pair in a set of $2N$ vertices is connected with probability p and the problem is to divide the $2N$ vertices into two sets V_1 and V_2 of N vertices each in such a way that the number of links between the two sets is minimal. FA showed that for an N-independent probability ($p = O(1)$) the bipartitioning problem bears close resemblance to the SK (Sherrington–Kirkpatrick) spin glass. In this letter we study a slightly modified version of their model in which the connectivity is fixed at every vertex (and not just on the average as in FA) and we show that for the case of extensive connectivity it has the same thermodynamic limit.

Solving the bipartitioning problem is equivalent to finding the ground state of the Hamiltonian

$$H' = -J \sum_{(j,l)} a_{jl} S_j S_l \tag{1}$$

while satisfying the additional condition

$$\sum_j S_j = 0. \tag{2}$$

S_j takes the values ± 1 and a_{jl} is 1 if the two vertices j and l are connected and 0 otherwise. FA used the distribution of the a_{jl}

$$a_{jl} = \begin{cases} 1 \\ 0 \end{cases} \qquad \text{with probability} \begin{cases} p \\ 1-p \end{cases} \tag{3}$$

where p is independent of N ($p = O(1)$). Here, we impose a stricter condition on the a_{jl} by demanding

$$\sum_{l(\neq j)} (a_{jl} - p) = 0 \tag{4}$$

for each j. Other than this restriction the a_{jl} are randomly 1 or 0.

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0305-4470/87/010009+03\$02.50

Equations (2) and (4) can be taken into account by adding terms to the effective Hamiltonian to give

$$H = H' + J_1 \sum_j \Big(\sum_{l(\neq j)} (a_{jl} - p) \Big)^2 + J_2 \Big(\sum_j S_j \Big)^2 . \tag{5}$$

Equations (2) and (4) are then satisfied by taking the limits $J_1 \to \infty$, $J_2 \to \infty$.

Following FA (to whom we refer for further details) we use the replica trick (Edwards and Anderson 1975, Sherrington and Kirkpatrick 1975) to find the free energy averaged over the distribution of the a_{jl}, the cost function being the zero temperature limit.

The J_1 term is linearised by a Hubbard-Stratonovich transformation and we obtain

$$\begin{aligned}
[Z^n]_{\mathrm{av}} = \mathrm{Tr}_S \int_{-\infty}^{\infty} & \Big[\prod_j \frac{\mathrm{d}u_j}{\sqrt{\pi\beta J_1}} \exp\Big(-\frac{1}{\beta J_1} \sum_j u_j^2 - 2(2N-1)\mathrm{i}pu_j \Big) \Big] \\
& \times \exp\Big(-\beta J_2 \sum_{\alpha=1}^{n} \Big(\sum_j S_j^\alpha \Big)^2 \Big) (1-p)^{N(2N-1)} \\
& \times \prod_{(j,l)} \Big[1 + \frac{p}{1-p} \exp\Big(\beta J \sum_{\alpha=1}^{n} S_j^\alpha S_l^\alpha + 2\mathrm{i}(u_j + u_l) \Big) \Big] .
\end{aligned} \tag{6}$$

$[\ldots]_{\mathrm{av}}$ denotes the average over the a_{jl}, the S_j^α carry a replica index and Tr_S is the trace over the states of the $2Nm$ 'spins'. Equation (6) replaces (3.3) in FA and reduces to that equation if all u_j are set equal to zero.

The logarithm of the last product in (6) can be expanded to give

$$\begin{aligned}
\ln \prod_{(j,l)} & \Big[1 + \frac{p}{1-p} \exp\Big(\beta J \sum_{\alpha=1}^{n} S_j^\alpha S_l^\alpha + 2\mathrm{i}(u_j + u_l) \Big) \Big] \\
& = \sum_{\lambda=0}^{\infty} (\beta J)^\lambda N^2 \frac{1}{\lambda !} \sum_{m=1}^{\infty} (-1)^{m-1} m^{\lambda-1} \Big(\frac{p}{1-p} \Big)^m \\
& \quad \times \sum_{\alpha_1, \ldots, \alpha_\lambda} \Big(\frac{1}{N} \sum_j \exp(2\mathrm{i}mu_j) S_j^{\alpha_1} \ldots S_j^{\alpha_\lambda} \Big)^2 .
\end{aligned} \tag{7}$$

In (7), setting $u_j = 0$ again reduces this to the case studied by FA (3.5) where the last square does not depend on m. In general, this m dependence considerably increases the difficulty of the problem and we look for a suitable approximation to simplify (7).

Until now, we have not discussed any possible N dependence of the coupling constants. FA argued that J has to be of order $N^{-1/2}$

$$J = N^{-1/2} \tilde{J} \tag{8}$$

to guarantee a sensible thermodynamic limit. An equivalent alternative procedure would be to introduce a corresponding scaling for $\beta = \tilde{\beta} N^{-1/2}$, the cost function being obtained in the limit $\tilde{\beta} \to \infty$, but we shall follow FA in applying scalings to J so that H is thermodynamically extensive. The number of terms in the Hamiltonian which depend on J_1 is proportional to N^3 and thus a sensible thermodynamic limit is obtained by

$$J_1 = N^{-2} \tilde{J}_1 . \tag{9}$$

For this choice of J_1, $\tilde{J}_1$ has to be taken to infinity (or at least large compared to $\tilde{J}$ and T) to satisfy (4). Each u_j integral in (6) now has the form

$$A_j = \int \frac{N \mathrm{d}u_j}{\sqrt{\pi \beta \tilde{J}_1}} \exp\Big(-\frac{N^2}{\beta \tilde{J}_1} u_j^2 - \mathrm{i}u_j [2(2N-1)p - 2M_j] \Big) \tag{10}$$

where M_j is the number of factors $\exp(2iu_j)$ appearing in a term in the final product in (6). Analytic continuation of the exponent into the complex plane gives a saddle point at

$$z_j = -i\frac{\beta\tilde{J}_1}{N^2}[(2N-1)p - M_j]. \tag{11}$$

Thus, for fixed (large) $\tilde{J}_1$ the method of steepest descent gives the dominant contribution to the integral from an area where z_j is of order $1/N$

$$z_j = N^{-1}\tilde{z}_j. \tag{12}$$

This justifies an expansion of $\exp(2iu_j)$ in u_j and the terms of leading order in $1/N$ are the same as in FA. This shows that effects due to the sharp constraint (4) are of higher order in $1/N$ and the Hamiltonian (5) can be mapped onto the SK spin glass in the same way in which FA did the mapping for their model.

Note, however, that the similarity between the model with fixed numbers of bonds at each vertex (which may well be different at different vertices) and the model where only the total number of bonds is given has only been shown for N-independent p. This analysis does not hold for a 'finite-valence' model where p is proportional to $1/N$ because J_1 is no longer proportional to N^{-2}. In that case differences between the two models can be expected.

Financial support from the Science and Engineering Research Council is gratefully acknowledged.

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